

On a property of 2-dimensional integral Euclidean lattices

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Abstract

Let Λ be any integral lattice in the 2-dimensional Euclidean space. Generalizing the earlier works of Hiroshi Maehara and others, we prove that for every integer $n > 0$, there is a circle in the plane \mathbb{R}^2 that passes through exactly n points of Λ .

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1 Introduction

We consider the following condition on 2-dimensional lattices $\Lambda \subset \mathbb{R}^2$.

Definition 1.1. If there is a circle in the plane \mathbb{R}^2 that passes through exactly n points of Λ for every integer $n > 0$, then Λ is called universally concyclic.

A lattice generated by $(a, b), (c, d) \in \mathbb{R}^2$, ($ad - bc \neq 0$) is denoted by $\Lambda[(a, b), (c, d)]$. In [4], Maehara introduced the term “universally concyclic”. Then, he and others showed the following results. In [5] and [3], Schinzel, Maehara and Matsumoto proved that \mathbb{Z}^2 , that is, $\Lambda[(1, 0), (0, 1)]$ is universally concyclic. Moreover let $a, b, c, d \in \mathbb{Z}$ be such that $q := ad - bc$ is a prime and $q \equiv 3 \pmod{4}$. Then $\Lambda[(a, b), (c, d)]$ is universally concyclic.

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The equilateral triangular lattice $\Lambda[(1, 0), (-1/2, \sqrt{-3}/2)]$ and rectangular lattice $\Lambda[(1, 0), (0, \sqrt{-3})]$ are universally concyclic.

Let $\mathbb{Z}[x] := \{a + bx \mid a, b \in \mathbb{Z}\}$. We remark that for a positive integer d , a lattice $\Lambda[(1, 0), (a, b\sqrt{-d})]$ is also given by $\mathbb{Z}[a + b\sqrt{-d}]$ in the complex plane. We define the set $A(k)$ as follows:

$$A(k) := \{z \in \mathbb{Z}[\sqrt{-3}] \mid |z|^2 = 7^k\}.$$

In [4], Maehara proved the following result:

Lemma 1.1 (cf. [4]). $\#A(k) = 2(k + 1)$.

Then, Maehara [4] proposed the following problems:

Problem 1.1 (cf. [4]). For every square-free integer $d > 1$ and a prime p such that $p = x^2 + y^2d$, we have $\#\{z \in \mathbb{Z}[\sqrt{-d}] \mid |z|^2 = p^k\} \geq 2(k + 1)$ for every k . Does equality always hold?

Problem 1.2 (cf. [4]). Is $\Lambda[(a, b), (c, d)]$ universally concyclic if $a, b, c, d \in \mathbb{Z}$ and $ad - bc \neq 0$.

Here, we answer Problems 1.1 and 1.2 affirmatively. In fact, we prove a slightly stronger assertion in Theorems 1.1 and 1.2 below. Let d be a square-free positive integer and K be the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. We define \mathcal{O}_K as the integer ring of K . Let $\mathbb{Z} \cdot a + \mathbb{Z} \cdot b$ denote the linear combination of a and b with integer coefficients. Then \mathcal{O}_K will be written as follows:

$$\mathcal{O}_K = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot w_K, \tag{1}$$

where

$$w_K = \begin{cases} \sqrt{-d} & \text{if } -d \equiv 2, 3 \pmod{4}, \\ \frac{-1 + \sqrt{-d}}{2} & \text{if } -d \equiv 1 \pmod{4}. \end{cases} \tag{2}$$

We denote by d_K the discriminant of K :

$$d_K = \begin{cases} -4d & \text{if } -d \equiv 2, 3 \pmod{4}, \\ -d & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

We review the concept of order in a quadratic field (for more details, see [2]). An order \mathcal{O} in a quadratic field K is a subset $\mathcal{O} \subset K$ such that

1. \mathcal{O} is a subring of K containing 1.
2. \mathcal{O} is a finitely generated \mathbb{Z} -module.

3. \mathcal{O} contains a \mathbb{Q} -basis of K .

We can now describe all orders in a quadratic fields:

Lemma 1.2 (cf. [2, page. 133]). *Let \mathcal{O} be an order in a quadratic field K of discriminant d_K . Then \mathcal{O} has a finite index in \mathcal{O}_K , and if we set $f = [\mathcal{O}_K : \mathcal{O}]$, then*

$$\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot fw_K, \quad (3)$$

where w_K is as in (2). Here f is called a conductor of the order \mathcal{O} .

We denote \mathcal{O} by \mathcal{O}_f if $f = [\mathcal{O}_K : \mathcal{O}]$. Now, we introduce the concept of proper ideals of an order. For any ideal \mathfrak{a} of \mathcal{O}_f , notice that

$$\mathcal{O}_f \subset \{\beta \in K \mid \beta\mathfrak{a} \subset \mathfrak{a}\}$$

since \mathfrak{a} is an ideal of \mathcal{O}_f . We say that an ideal \mathfrak{a} of \mathcal{O}_f is proper whenever equality holds, i.e., when

$$\mathcal{O}_f = \{\beta \in K \mid \beta\mathfrak{a} \subset \mathfrak{a}\}.$$

A quadratic form F is called integral if all the coefficients of F are rational integers. A lattice Λ is called integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in \Lambda$, where (x, y) is the standard inner product. Generally, it is well-known that there exists a one-to-one correspondence between the set of proper ideal classes of the order \mathcal{O}_f and the equivalence class of primitive positive definite integral quadratic forms $F(x, y)$ with discriminant $f^2 d_K < 0$ (see Theorem 2.2 in Section 2, [1, Chapter 2, §7-6], [6, §11]). Hence, we consider the proper ideal classes of \mathcal{O}_f to be the lattice in \mathbb{R}^2 corresponding to a quadratic forms $F(x, y)$. On the other hand, any 2-dimensional integral Euclidean lattice can be considered as some proper ideal class of \mathcal{O}_f . We define Λ as the proper ideal classes of \mathcal{O}_f . Then, we prove the following theorems:

Theorem 1.1. *Let $n \in \mathbb{N}$ and assume that $n \neq 1$. Let p be a prime number such that there exists a $z \in \mathbb{Z}[\sqrt{-n}]$ with $|z|^2 = p$, $\left(\frac{d_K}{p}\right) = 1$ and $(p, f) = 1$, where (\cdot) is the Legendre symbol. Then,*

$$\#\{z \in \mathbb{Z}[\sqrt{-n}] \mid |z|^2 = p^k\} = 2(k+1).$$

Theorem 1.2. *All the 2-dimensional integral lattices in \mathbb{R}^2 are universally concyclic.*

Remark 1.1. We remark that there exist some non-integral lattices which are not universally concyclic. Maehara also proved in [4] that if τ is a transcendental number, then $\Lambda[(1, \tau), (0, 1)]$ cannot contain four concyclic points, hence is not universally concyclic. The rectangular lattice $\Lambda[(\alpha, 0), (0, \beta)]$ does not contain five concyclic points if and only if $(\alpha/\beta)^2$ is an irrational number. Hence, some additional integrality conditions are necessary to ensure this property.

2 Preliminaries

In this paper, we consider the 2-dimensional integral Euclidean lattices. We shall always assume that d denotes a positive square-free integer. Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and let \mathcal{O}_K be its ring of algebraic integers defined by (1). As we mentioned in Section 1, there exists a one-to-one correspondence between the set of fractional ideal classes of the unique quadratic field $\mathbb{Q}(\sqrt{-d})$ and the equivalence class of primitive positive definite integral quadratic forms $F(x, y)$ with discriminant $d_K < 0$ [6, §10]. More generally, there exists a one-to-one correspondence between the set of fractional proper ideal classes of order \mathcal{O}_f and the equivalence class of primitive positive definite integral quadratic forms $F(x, y)$ with discriminant $f^2 d_K < 0$ [1, Chapter 2, §7-6], [6, §11]. We remark that the value $f^2 d_K$ is called the discriminant of the order \mathcal{O}_f . Finally, we give the well-known theorems needed later.

Theorem 2.1 (cf. [2, page 104]). *We can classify prime ideals of a quadratic field as follows:*

1. *If p is an odd prime and $\left(\frac{d_K}{p}\right) = 1$ (resp. $d_K \equiv 1 \pmod{8}$) then*

$$(p) = \mathfrak{p}\mathfrak{p}' \text{ (resp. } (2) = \mathfrak{p}\mathfrak{p}'),$$

where \mathfrak{p} and \mathfrak{p}' are prime ideals with $\mathfrak{p} \neq \mathfrak{p}'$, $N(\mathfrak{p}) = N(\mathfrak{p}') = p$ (resp. $N(\mathfrak{p}) = 2$).

2. *If p is an odd prime and $\left(\frac{d_K}{p}\right) = -1$ (resp. $d_K \equiv 5 \pmod{8}$) then*

$$(p) = \mathfrak{p} \text{ (resp. } (2) = \mathfrak{p}),$$

where \mathfrak{p} is a prime ideal with $N(\mathfrak{p}) = p^2$ (resp. $N(\mathfrak{p}) = 4$).

3. *If $p \mid d_K$ then*

$$(p) = \mathfrak{p}^2,$$

where \mathfrak{p} is a prime ideal with $N(\mathfrak{p}) = p$.

Theorem 2.2 (cf. [2, Theorem 7.7]). *Let \mathcal{O} be an order of discriminant D in an imaginary quadratic field K .*

1. *If $F(x, y) = ax^2 + bxy + cy^2$ is a primitive positive definite integral quadratic form of discriminant D , then $[a, (-b + \sqrt{D})/2]$ is a proper ideal of \mathcal{O} .*
2. *The map sending $F(x, y)$ to $[a, (-b + \sqrt{D})/2]$ induces an isomorphism between the form class group and the ideal class group.*
3. *A positive integer m is represented by a form $F(x, y)$ if and only if m is the norm $N(\mathfrak{a})$ of some ideal \mathfrak{a} in the corresponding ideal class mentioned in 2.*

Lemma 2.1 (cf. [2, Lemma 7.18]). *Let \mathcal{O}_f be an order of conductor f . We say that a non-zero \mathcal{O}_f -ideal \mathfrak{a} is prime to f provided that $\mathfrak{a} + f\mathcal{O}_f = \mathcal{O}_f$.*

1. *An \mathcal{O}_f -ideal \mathfrak{a} is prime to f if and only if its norm $N(\mathfrak{a})$ is relatively prime to f .*
2. *Every \mathcal{O}_f -ideal prime to f is proper.*

Proposition 2.1 (cf. [2, Proposition 7.20]). *Let \mathcal{O}_f be an order of conductor f in an imaginary quadratic field K . We say that a non-zero \mathcal{O}_K -ideal \mathfrak{a} is prime to f provided that $\mathfrak{a} + f\mathcal{O}_K = \mathcal{O}_K$. If \mathfrak{a} is an \mathcal{O}_K -ideal prime to f , then $\mathfrak{a} \cap \mathcal{O}_f$ is an \mathcal{O}_f -ideal prime to f of the same norm.*

Proposition 2.2 (cf. [2, Exercise 7.26]). *Let \mathcal{O}_f be an order of conductor f . Then \mathcal{O}_f -ideals prime to the conductor can be factored uniquely into prime \mathcal{O}_f -ideals (which are also prime to f).*

Theorem 2.3 (cf. [2, Theorem 9.4]). *Let $n > 0$ be an integer, and L be the ring class field of the order $\mathbb{Z}[\sqrt{-n}]$ in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-n})$. If p is an odd prime not dividing n , then*

$$p = x^2 + ny^2 \Leftrightarrow p \text{ splits completely in } L.$$

3 Proof of Theorem 1.1

Proof of Theorem 1.1. We remark that $\mathbb{Z}[\sqrt{-n}]$ can be considered as the order $\mathbb{Z}[\sqrt{-n}] = \mathcal{O}_f \subset K = \mathbb{Q}(\sqrt{-d})$ for some f and d with the following condition $-4n = f^2 d_K$, namely,

$$n = \begin{cases} f^2 d & \text{if } -d \equiv 2, 3 \pmod{4}, \\ \frac{f^2 d}{4} & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

Therefore, we remark that $\mathbb{Z}[\sqrt{-n}] = \mathcal{O}_f$.

We fix a prime p such that there exists a $z \in \mathbb{Z}[\sqrt{-n}]$ with $|z|^2 = p$, $\left(\frac{d_K}{p}\right) = 1$ and $(p, f) = 1$. Because of Theorem 2.1, $(p) = \mathfrak{p}\mathfrak{p}'$ in \mathcal{O}_K for some \mathfrak{p} . Moreover, the condition $z \in \mathbb{Z}[\sqrt{-n}]$ implies that the ideals \mathfrak{p} and \mathfrak{p}' are principal ideals. We set

$$\begin{aligned}\mathfrak{q} &= \mathfrak{p} \cap \mathcal{O}_f \\ \mathfrak{q}' &= \mathfrak{p}' \cap \mathcal{O}_f.\end{aligned}$$

Then, by Proposition 2.1, the ideals \mathfrak{q} and \mathfrak{q}' are principal ideals of \mathcal{O}_f prime to f . Because of Lemma 2.1, \mathcal{O}_f -ideal prime to f is proper and using the unique factorization of proper ideals in Proposition 2.2, the ideals of norm p^k are as follows:

$$\mathfrak{q}^k, \mathfrak{q}^{k-1}\mathfrak{q}', \dots, \mathfrak{q}'^k. \quad (4)$$

Let z_1 be the element of $\mathbb{Z}[\sqrt{-n}]$ with norm p^k . Because of Lemma 2.1, (z_1) is a proper \mathcal{O}_f -ideal. Moreover, for $-z_1 \in \mathbb{Z}[\sqrt{-n}]$, the ideals (z_1) and $(-z_1)$ are same proper \mathcal{O}_f -ideals. Hence, there exists a one-to-one correspondence between the non-equivalent elements of $\mathbb{Z}[\sqrt{-n}]$ with norm p^k under the action of $\{\pm 1\}$ and the set of proper \mathcal{O}_f -ideals of norm p^k defined by (4). This completes the proof of Theorem 1.1. \square

4 Proof of Theorem 1.2

4.1 Setup

Proposition 4.1. *For any positive integers n and a , there exists a prime p prime to n such that*

$$p = x^2 + ny^2$$

with $y \equiv 0 \pmod{4a}$.

Proof. We set $n' = 16a^2n$. Let L be the ring class field of the order $\mathbb{Z}[\sqrt{-n}]$. (We refer to Cox [2] for the concept of ring class fields.) Because of Theorem 2.3, there exists a prime p such that

$$\begin{aligned}p &= x^2 + n'y^2 \\ &= x^2 + n(4ay)^2\end{aligned}$$

if and only if p splits completely in L . Then the primes that split completely in L have density $1/[L : K]$, and in particular there are infinitely many of them (cf. [2, Corollary 5.21] and [2, Corollary 8.18]). Hence, there exists a prime p prime to n . Therefore, we complete the proof of Proposition 4.1. \square

Because of Proposition 4.1, there exists prime p prime to n such that $p = x_1^2 + ny_1^2$ with $y_1 \equiv 0 \pmod{4a}$. We fix such a prime and denote it by $p_{n,a}$. Then we define $A_{n,a}(k)$ as follows:

$$A_{n,a}(k) := \{z \in \mathbb{Z}[\sqrt{-n}] \mid |z|^2 = p_{n,a}^k\}.$$

By Proposition 4.1, if $x + y\sqrt{-n} \in A_{n,a}(k)$ then $y \equiv 0 \pmod{4a}$ and

$$x + y \equiv \pm j \pmod{4a}, \quad (5)$$

where $j \equiv x_1^k \pmod{4a}$, $1 \leq j \leq 4a - 1$. So, we define $\check{A}_{n,a}(k)$ as follows:

$$\check{A}_{n,a}(k) := \{x + y\sqrt{-n} \in A_{n,a}(k) \mid x + y \equiv -j \pmod{4a}\}.$$

Lemma 4.1. $\#A_{n,a}(k) = 2(k+1)$ and $\#\check{A}_{n,a}(k) = k+1$.

Proof. Because of Proposition 4.1, $(d_K/p_{n,a}) = 1$ and $(p_{n,a}, f) = 1$. Hence, by Theorem 1.1 $\#A_{n,a}(k) = 2(k+1)$. If $x + y\sqrt{-n} \in A_{n,a}(k)$, then $x \neq 0$, $-x + y\sqrt{-n} \in A_{n,a}(k)$, and only one of them belongs to $\check{A}_{n,a}(k)$. Therefore, $\#\check{A}_{n,a}(k) = k+1$. \square

4.2 Proof of Theorem 1.2

Here, we start the proof of Theorem 1.2.

Proof of Theorem 1.2. Let Λ be a 2-dimensional integral lattice and let the associated quadratic form be $ax^2 + bxy + cy^2$. Let $\mathcal{O}_f \subset \mathbb{Q}\sqrt{-d}$ be the order corresponding to the lattice Λ . We set $n = -f^2d_K$ and $\alpha := (-b + \sqrt{-n})/(2\sqrt{a})$. It is enough to show that for each integer $k > 0$, there is a circle in the complex plane that passes through exactly $k+1$ points of Λ . For $k > 0$, define a circle Γ_k in complex plane as follows:

$$|4\sqrt{a}z - j|^2 = p_{n,a}^k,$$

where j is defined by (5). Let $C(k)$ be the subset of Λ lying on the circle Γ_k . We show that $\#C(k) = k+1$. If $z = \sqrt{a}x + \alpha y \in C(k)$ then $4\sqrt{a}z - j = 4ax - 2by - j + 2y\sqrt{-n}$, so $4ax - 2by - j + 2y \equiv -j \pmod{4a}$. Therefore $4\sqrt{a}z - j \in \check{A}_{n,a}(k)$. Hence we can define the map $\varphi : C(k) \rightarrow \check{A}_{n,a}(k)$ by:

$$z \mapsto 4\sqrt{a}z - j.$$

This map is a bijection. To see this, suppose $x + y\sqrt{-n} \in \check{A}_{n,a}(k)$. Then $x + y \equiv -j \pmod{4a}$, that is, $x + by + j \equiv 0 \pmod{4a}$. Moreover, by

Proposition 4.1, $y \equiv 0 \pmod{4a}$, and hence y is even. Therefore, we can define a map from $\check{A}_{n,a}(k)$ to $C(k)$ as follows:

$$x + y\sqrt{-n} \mapsto \frac{x + by + j}{4\sqrt{a}} + \frac{y}{2}\alpha.$$

This gives the inverse of φ . Therefore φ is surjective, that is, $\sharp C(k) = \sharp \check{A}_{n,a}(k) = k + 1$. \square

Informing Hiroshi Maehara of Theorem 1.2, he proved the following fact:

Corollary 4.1. *If $(\alpha/\beta)^2 \in \mathbb{Q}$ then $\Lambda[(\alpha, 0), (0, \beta)]$ is universally concyclic.*

Proof. We assume that $(\alpha/\beta)^2 = b/a$, where b/a is irreducible fraction. Then, the lattices $\Lambda[(\alpha, 0), (0, \beta)]$ and $\Lambda[(a, 0), (0, \sqrt{ab})]$ are similar under the similarity transformation α/a and $\Lambda[(a, 0), (0, \sqrt{ab})]$ is integral lattice. Because of Theorem 1.2, $\Lambda[(a, 0), (0, \sqrt{ab})]$ is universally concyclic, so is $\Lambda[(\alpha, 0), (0, \beta)]$. \square

Remark 4.1. Finally, we generalize the definition of universally concyclic to higher dimensions.

Definition 4.1. Let $\Lambda \subset \mathbb{R}^d$ be a d -dimensional lattice. If there is a spherical surface S^{d-1} in \mathbb{R}^d that passes through exactly n points of Λ for every integer $n > 0$, then Λ is called universally concyclic.

In [4], Maehara remark that \mathbb{Z}^3 is universally concyclic because the spherical surface $(4x-1)^2 + (4y)^2 + (4z-\sqrt{2})^2 = 17k+2$ passes through exactly $k+1$ points of \mathbb{Z}^3 . We also remark that any integral lattices in higher dimension \mathbb{R}^d are universally concyclic:

Corollary 4.2. *Any integral lattices in \mathbb{R}^d are universally concyclic.*

Proof. Let Λ be an integral lattice in \mathbb{R}^d . We define sublattices $\{\Lambda^{(i)}\}_{i=2}^d$ such that

$$\Lambda^{(2)} \subset \Lambda^{(3)} \subset \dots \subset \Lambda^{(d)} = \Lambda$$

and $\Lambda^{(i)}$ spans \mathbb{R}^i which we denote by $\mathbb{R}^{(i)}$ for all i . Because of Theorem 1.2, for each $k > 0$, we can define the circle $S^{(1)} \subset \mathbb{R}^{(2)}$ that passes through exactly k points of $\Lambda^{(2)}$.

Let $O^{(1)}$ be the center of $S^{(1)}$ and let ℓ be a half line in $\mathbb{R}^{(3)}$ whose origin is $O^{(1)}$, which is orthogonal to $\mathbb{R}^{(2)}$. We define the sphere $S^{(2)}(a)$, whose center $O^{(2)}(a)$ lies on ℓ , the distance between $O^{(1)}$ and $O^{(2)}(a)$ is a and whose radius is $\sqrt{a^2 + (\text{radius of } S^{(1)})^2}$. We assume that $0 \leq a \leq 1$.

Since Λ is an integral lattice, the number of the points of $\Lambda^{(3)}$ which intersect in $S^{(1)}(a)$ is finite for any $0 \leq a \leq 1$. Moreover, for $a_1 \neq a_2$,

the intersection of $S^{(1)}(a_1)$ and $S^{(1)}(a_2)$ is the points of $\Lambda^{(2)}$ in Λ , namely, the points of $S^{(1)}$. On the other hand, for $0 \leq a \leq 1$, the number of the spheres $S^{(2)}(a)$ is infinite. Therefore, there exists a number a_0 such that the intersection of $S^{(2)}(a_0)$ and Λ is the points of $\Lambda^{(2)}$. We denote $S^{(2)}(a)$ by $S^{(2)}$ and $S^{(2)}$ passes through exactly k points of $\Lambda^{(3)}$. We can define the spheres $S^{(3)}, \dots, S^{(d-1)}$ recursively such that each of $\{S^{(i)}\}_{i=3}^{d-1}$ passes through exactly k points of Λ , as we defined $S^{(2)}$ in $\mathbb{R}^{(3)}$. \square

So, we have shown that any integral lattices in \mathbb{R}^d are universally concyclic. However, the points of lattice lying on the sphere constructed in the proof of Corollary 4.2 are on the plane $x_3 = \dots = x_d = 0$. Hence, Maehara added some conditions to Definition 4.1 and showed the following theorem:

Theorem 4.1 (cf. [4]). *For $n > d \geq 2$, there is a sphere in \mathbb{R}^d that passes through exactly n lattice points on \mathbb{Z}^d , and moreover, the n lattice points span a d -dimensional polytope.*

Therefore, we can state the following problem:

Problem 4.1. Let Λ be an integral lattice in \mathbb{R}^d . We assume $n > d \geq 2$. Is there a sphere in \mathbb{R}^d that passes through exactly n lattice points on Λ , which span a d -dimensional polytope?

A set of points in the d -dimensional Euclidean space is said to be in general position if no $d+1$ of them lie in a $(d-1)$ -dimensional plane. Then, Maehara also proposed the following problem:

Problem 4.2 (cf. [4]). Is there a sphere in \mathbb{R}^3 that passes through a given number of lattice points in general position on \mathbb{Z}^3 ?

It is also an interesting open problem to prove or disprove a similar conclusion as in Problem 4.2 for any integral lattices in higher dimension \mathbb{R}^d .

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